B-closed Spaces and Fuzzy b-closed Spaces

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Abstract: The purpose of this paper is to establish and project the theorems which exhibit the characterization of b-closed spaces and obtain some of interesting properties of b-closed spaces. Moreover, fuzzy b-closed spaces are introduced, and some characterization of their properties are obtained.

Keywords: Topological Spaces; b-Closed Spaces; Fuzzy Spaces; Fuzzy b-Closed Spaces.

1. Introduction

In [4], the authors introduced the notion of b-closed spaces and investigated its fundamental properties. The concept of b-open sets in fuzzy settings was introduced by Benchalli and Karnel [1]. In this paper, we investigate a class of sets called b- closed sets. We study some of its basic properties. Afterward, we introduce the concept of fuzzy b-closed spaces.

In particular, the notion of generalized b-closed spaces and its various characterizations are given (see Section 2). In Section 3, we study various forms of fuzzy b-closed spaces.

Now, we recall the following definitions which are useful in the sequel.

Proposition 1.1. A subset $A$ of a space $X$ is b-open if and only if $A = B \cup C$, where $B$ is semi-open and $C$ is preopen.

Proposition 1.2. (i) Let $A$ and $B$ be subsets of a space $X$ such that $A \subseteq B$. If $A \in bo(X)$, then $A \in bo(B)$.

(ii) If $A \in bo(B)$, $B \in ao(X)$, then $A \in bo(X)$.

Proposition 1.3. A space $X$ is extremally disconnected if and only if every b-open subset of $X$ is preopen.

Proposition 1.4. A space $X$ is strongly irresolvable if and only if every b-open subset of $X$ is semi-open.

Proposition 1.5. For a space $X$, the following are equivalent:

(i) $X$ is locally indiscrete,

(ii) Every b-open subset of $X$ is preclosed.

2. b-closed Spaces

Definition 2.1. A space $X$ is called b-closed if any b-open cover of $X$ has a finite subfamily, the union of the preclosures of whose members covers $X$.

Remark 2.2. Since $so(X) \cup po(X) \subseteq bo(X)$, and since $pclA = \overline{A}$ whenever $A$ is semi-open, it is clear that every b-closed space is both S-closed and p-closed. However, the author asks about the existence of a space that is both S-closed and p-closed but not b-closed.

The following two propositions follows from Propositions 1.3 and 1.4 and from the fact that $pclA = \overline{A}$ whenever $A$ is semi-open.

Proposition 2.3. For an extremally disconnected space $X$, the following are equivalent:

(i) $X$ is b-closed.

(ii) $X$ is p-closed.

Proposition 2.4. For a strongly irresolvable space $X$, the following are equivalent:

(i) $X$ is b-closed.

(ii) $X$ is S-closed.

The following result is an immediate consequence of Proposition 1.1 and from the fact that $so(X) \cup po(X) \subseteq bo(X)$.

Proposition 2.5. A space $X$ is b-closed if and only if any cover of $X$ whose members are semi-open or preopen has a finite subfamily, the union of the preclosures of whose members covers $X$.

Lemma 2.6. A subset $A$ of a space $X$ is b-open if and only if there exists a preopen subset $U$ of $X$ such that $U \subseteq A \subseteq pclU$.

Theorem 2.7. For a space $X$, the following are equivalent:

(i) $X$ is b-closed.

(ii) Any regular p-open cover of $X$ has a finite subfamily, the union of the preclosures of whose members covers $X$.

(iii) Any pre-regular p-closed cover of $X$ has a finite subcover.

Proof. (i) to (ii): Follows since every regular p-open set is b-open.

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(ii) to (iii): Follows since every pre-regular $p$-closed set is regular $p$-open and preclosed.

(iii) to (i): Let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a b-open cover of $X$.

Then by Lemma 2.6, for each $\alpha \in \Lambda$, there exists a preopen subset $V_\alpha$ of $X$ such that $V_\alpha \subseteq U_\alpha \subseteq pclV_\alpha$. Now $V = \{pclV_\alpha : \alpha \in \Lambda\}$ is preregular $p$-closed cover of $X$ and thus by (ii), there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $X = \bigcup_{i=1}^n pclV_{\alpha_i} = \bigcup_{i=1}^n pclU_{\alpha_i}$. Hence, $X$ is b-closed.

The following result follows from the definition of a b-closed space and from Propositions 2.5 and Theorem 2.7, the straightforward proof is omitted.

**Proposition 2.8.** For a space $X$, the following are equivalent:

(i) $X$ is b-closed.

(ii) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of b-closed subsets of $X$ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \emptyset$.

(iii) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of regular $p$-closed subsets of $X$ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \emptyset$.

(iv) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of pre-regular $p$-open subsets of $X$ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcap = \{\text{pint}U_\alpha : \alpha \in \Lambda_0\} = \emptyset$.

(v) For any family $u = \{U_\alpha : \alpha \in \Lambda\}$ of pre-regular $p$-open subsets of $X$ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcap = \{U_\alpha : \alpha \in \Lambda_0\} = \emptyset$.

**Definition 2.9.** Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be a b-pre-$\theta$-accumulation point of $A$ if $pcl(U) \cap A \neq \emptyset$ for every b-open subset $U$ of $X$ that contains $X$. The set of all b-$\theta$-accumulation points of $A$ is called the b-pre-$\theta$-closure of $A$ and is denoted by $b-precl\theta(A)$. $A$ is said to be b-pre-$\theta$-closed if $b-precl\theta(A) = A$. The complement of a b-pre-$\theta$-closed set is called b-pre-$\theta$-open.

It is clear that $A$ is b-pre-$\theta$-open if and only if for each $x \in A$, there exists a b-open set $U$ such that $x \in U \subseteq pclU \subseteq A$, thus, every b-pre-$\theta$-open set is b-open.

**Definition 2.10.**

(i) A space $X$ is called b-regular if for each b-open subset $U$ of $X$ and for each $x \in U$ there exists a b-open subset $V$ of $X$ and a b-closed subset $F$ of $X$ such that $x \in V \subseteq F \subseteq U$.

(ii) A space $X$ is called strongly b-regular if for each b-open subset $U$ of $X$ and for each $x \in U$ there exists a b-closed subset $V$ of $X$ and a b-pre-regular $p$-closed subset $F$ of $X$ such that $x \in V \subseteq F \subseteq U$.

The following lemma can be easily established.

**Lemma 2.11.**

(i) A space $X$ is strongly b-regular if and only if every b-open subset of $X$ is b-pre-$\theta$-open.

(ii) If $A$ is pre-regular $p$-open, then $A$ is b-pre-$\theta$-closed.

(iii) $bclA \subseteq bcl\theta A$.

(iv) If $A$ is preopen, then $bcl\theta A = bclA$.

**Remark 2.12.**

(i) The converse of Lemma 2.11 (ii) is not true, e.g. if $X$ is an infinite set and $\tau_{cof}$ is the cofinite topology on $X$, then in $(X, \tau_{cof})$, every cofinite subset of $X$ is b-pre-$\theta$-closed but not pre-regular $p$-closed as it is not preclosed (observe that the nonempty b-open (preopen) subsets of $(X, \tau_{cof})$ are the infinite subsets of $X$).

It follows also from Proposition 1.5 that every locally indiscrete space is strongly b-regular. The converse is, however, not true, e.g. if $X$ is an infinite set and $\tau_{cof}$ is the cofinite topology on $X$, then in $(X, \tau_{cof})$, every b-open subset of $X$ is b-pre-$\theta$-open. Thus by Proposition 2.11 (i), $X$ is strongly b-regular. However, $(X, \tau_{cof})$ is not locally indiscrete.

**Theorem 2.13.** A space $X$ is b-closed if and only if every b-pre-$\theta$-open cover of $X$ has a finite subcover.

**Proof.** Suppose that $X$ is b-closed and let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a b-pre-$\theta$-open cover of $X$. Then for each $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in U_{\alpha_x}$. Since $U_{\alpha_x}$ is b-pre-$\theta$-open, there exists a b-open set $V_x$ such that $x \in V_x \subseteq pclV_x \subseteq U_{\alpha_x}$, but $X$ is b-closed, so there exists $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$.

Sufficiency. follows from Theorem 2.7 and Lemma 2.11 (ii).

**Proposition 2.14.** Let $X$ be a b-closed, strongly b-regular space. Then $X$ is finite.

**Proof.** It follows from Lemma 2.11 (i) and Theorem 2.13, that if $X$ is a B-closed, strongly b-regular space, then every b-open cover of $X$ has a finite subcover. Since
such that the two neighborhood of \( \mathcal{U} \), and \( \{ x \} \). Since \( \mathcal{U} = \emptyset \) and \( \{ x \} \) is said to be,

\[ (\mu_1 + \mu_2)(x) = \bigvee \{ \mu_1(x_1) \wedge \mu_2(x_2) \mid x = x_1 + x_2 \}. \]

And for a scalar \( t \) of \( K \) and a fuzzy subset \( \mu \) of \( X \), the fuzzy subset \( t\mu \) is defined by

\[ (t\mu)(x) = \begin{cases} \frac{\mu(x)}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \text{ and } x \neq 0 \\ \vee \{ \mu(y) \mid y \in X \} & \text{if } t = 0 \text{ and } x = 0 \end{cases} \]

for every \( t \in K \). Let \( \mu \in I^X \) be said to be,

1. convex if \( t\mu + (1-t)\mu \leq \mu \) for each \( t \in [0,1] \)
2. balanced if \( t\mu \leq \mu \) for each \( t \in K \) with \( |t| \leq 1 \)
3. absorbing if \( \vee \{ t\mu(x) \mid |t| > 0 \} = 1 \) for all \( x \in X \).

Definition 3.3. [5] Let \( (X, \tau) \) be a topological space and \( \omega(\tau) = \{ f : (X, \tau) \to [0,1] \mid \tau \text{ is lower semicontinuous} \} \),

then \( \omega(\tau) \) is a fuzzy topology on \( X \). This topology is called the fuzzy topology generated by \( \tau \) on \( X \). The fuzzy usual topology on \( K \) means the fuzzy topology generated by the usual topology of \( K \).

\[ n \geq M \text{ implies } \frac{t}{2} \rho(x_n - x) > 1 - \varepsilon \]

therefore

\[ n \geq M \text{ implies } P_{t,\varepsilon}(x_n - x) \leq \frac{t}{2} < t \]

Definition 3.4. [5] A fuzzy linear topology on a vector space \( X \) over \( K \) is a fuzzy topology on \( X \) such that the two mappings

\[ + : X \times X \to X, \quad (x, y) \mapsto x + y \]

\[ \cdot : K \times X \to X, \quad (t, x) \mapsto tx \]

are continuous when \( K \) has the fuzzy usual topology and \( K \times X \) and \( X \times X \) have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a fuzzy linear space or a fuzzy topological vector space.

Definition 3.5. [5] Let \( x \) be a point in a fuzzy topological space \( X \). A family \( F \) of neighborhood of \( x \) is called a base for the system of all neighborhoods of \( x \) if for each neighborhood \( \mu \) of \( x \) and each \( 0 < \theta < \mu(x) \), there exists \( \mu_0 \in F \) with \( \mu_0 \leq \mu \) and \( \mu_0(x) > \theta \).

Definition 3.6. [6] A fuzzy semi norm on \( X \) is a fuzzy set \( \rho \) in \( X \) which is convex, balanced and absorbing. If in addition

\[ \wedge \{ (t \rho)(x) \mid t > 0 \} \]

for \( x \neq 0 \), then \( \rho \) is called a fuzzy norm.
Definition 3.7. [6] If $\rho$ is a fuzzy semi norm on $X$, then the family $B_{\rho} = \{ \theta \land (t \rho) \mid 0 < \theta \leq 1, t > 0 \}$ is a base at zero for a fuzzy linear topology $\tau_{\rho}$. The fuzzy topology $\tau_{\rho}$ is called the fuzzy topology induced by the fuzzy semi norm $\rho$. And a linear space equipped with a fuzzy semi norm is called a fuzzy semi normed linear space.

Definition 3.8. [8] Let $\rho$ be a fuzzy semi norm on $X$. $P_{\rho} : X \to R_{+}$ is defined by

\[ P_{\rho}(x) = \land \{ t > 0 \mid t \rho(x) > \varepsilon \} \]

For each $\varepsilon \in (0,1)$. 

Theorem 3.9. [8] The $P_{\rho}$ is a semi norm on $X$ for each $\varepsilon \in (0,1)$. Further $P_{\rho}$ is norm on $X$ for each $\varepsilon \in (0,1)$ if and only if $\rho$ is a fuzzy norm on $X$.

Definition 3.10. A fts $X$ is said to be fuzzy b-closed iff for every family $\lambda$ of fuzzy b-open set such that $\nbigcup_{\lambda} A = 1_{x}$ there is a finite subfamily $\delta \subseteq \lambda$ such that that

\[ \nbigvee_{A \in \delta} bCl(A)(x) = 1_{x} \]

for every $x \in X$.

Definition 3.11. A fuzzy set $U$ in a fts $X$ is said to be fuzzy b-closed relative to $X$ iff for every family $\lambda$ of fuzzy b-open set such that $\nbigvee A = 1_{x}$ there is a finite subfamily $\delta \subseteq \lambda$ such that

\[ \nbigvee_{A \in \delta} bCl(A)(x) = U(x) \]

for every $x \in S(U)$.

Remark 3.12. Every fuzzy b-compact space is fuzzy b-closed, but the converse is not true.

Theorem 3.13. A fts $X$ is fuzzy b-closed iff for every fuzzy filterbases $\Gamma$ in $X,

\[ \nbigwedge_{G \in \Gamma} bCl(G) \neq 0_{x} \]

Proof. Let $\mu$ be a fuzzy b-open set cover of $X$ and let for every finite family of $\mu$, $\nbigvee_{A \in \mu} bCl(A)(x) < 1_{x}$ for some $x \in X$. Then $\nbigwedge_{G \in \Gamma} bCl(G)(x) > 0_{x}$ for some $x \in X$. Thus $\nbigwedge_{G \in \Gamma} bCl(G) = \Gamma$ forms a fuzzy b-open filterbases in $X$. Since $\mu$ is a fuzzy b-open set cover of $X$, then

\[ \nbigwedge_{A \in \mu} A = 0_{x} \]

which implies

\[ \nbigwedge_{A \in \mu} bCl(G) = 0_{x} \]

which is a contradiction. Then every fuzzy b-open $\mu$ of $X$ has a finite subfamily $\delta$ such that

\[ \nbigvee_{A \in \delta} bCl(A)(x) = 1_{x} \]

for every $x \in X$.

Hence $X$ is fuzzy b-closed.

Conversely, suppose there exists a fuzzy b-open filterbases $\Gamma$ in $X$ such that

\[ \nbigvee_{G \in \Gamma} bCl(G) = 0_{x} \]

That implies

\[ \nbigvee_{G \in \Gamma} (bCl(G)(x)) = 1_{x} \]

for $x \in X$ and hence

\[ \mu = \{ bCl(G) \mid G \in \Gamma \} \]

is a fuzzy b-open set cover of $X$.

Since $X$ is fuzzy b-closed, by definition $\mu$ has a finite subfamily $\delta$ such that

\[ \nbigvee_{G \in \delta} bCl(G)(x) = 1_{x} \]

for every $x \in X$, and hence

\[ \nbigwedge_{G \in \delta} bCl(G) = 0_{x} \]

is a contradiction. Hence $\nbigwedge_{G \in \Gamma} bCl(G) \neq 0_{x}$.

Theorem 3.14. Let $f : (X,\tau) \to (Y,\sigma)$ be a fuzzy $b$–continuous surjection. If $X$ is fuzzy b-closed space, then $Y$ is fuzzy b-closed space.

Proof. Let $\{ A_{\lambda} : \lambda \in \Lambda \}$ be a fuzzy b-open cover of $Y$. Since $f$ is fuzzy b–continuous, $\{ f^{-1}(A_{\lambda}) : \lambda \in \Lambda \}$ is fuzzy b-open cover of $X$. By hypothesis, there exists a finite subset $\Delta$ of $\Gamma$ such that $\nbigvee_{\lambda \in \Delta} bCl(f^{-1}(A_{\lambda})) = 1_{x}$. Since $f$ is surjection and by theorem

\[ I_{Y} = f(1_{x}) = f(\nbigvee_{\lambda \in \Delta} bCl(f^{-1}(A_{\lambda}))) \]

\[ \leq \nbigvee_{\lambda \in \Delta} bCl(f(f^{-1}(A_{\lambda})) = \nbigvee_{\lambda \in \Delta} bCl(A_{\lambda})) \]

Hence $Y$ is fuzzy b-closed space.

References


